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TECHNICAL NOTE

The Equivalence of Three Shear–Normal Stress Forms of the Hoek–Brown Criterion

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List of Symbols

c Instantaneous cohesive strength at point $(\sigma_\alpha, \tau_\alpha)$ in Bray and Ucar equation
h Parameter in Bray, Ucar, and Londe equation
m, s Dimensionless empirical constant in Hoek–Brown criterion
C Instantaneous cohesive strength in Londe equation
N Dimensionless normal stress in Londe equation
N_1 $= \sigma_1 / m\sigma_c + s / m^2$, dimensionless major principal stress in Londe equation
N_3 $= \sigma_3 / m\sigma_c + s / m^2$, dimensionless minor principal stress in Londe equation
T Dimensionless shear stress in Londe equation
β Instantaneous friction angle at point $(\sigma_\alpha, \tau_\alpha)$ in Bray and Ucar equation
ϕ Instantaneous friction angle at point (N, T) in Londe equation
σ Normal stress
σ_1 Major principal stress
σ_3 Minor principal stress
σ_c Uniaxial compressive strength of intact rock
σ_α Normal stress in Bray and Ucar equation
τ Shear stress
τ_α Shear strength corresponding to normal stress σ_α in Bray and Ucar equation
τ'_α $= \tan\beta$, gradient of the tangent at point $(\sigma_\alpha, \tau_\alpha)$ in Bray and Ucar equation

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1 Introduction

The empirical Hoek–Brown criterion, because of its simplicity and ability to describe the nonlinear behaviour of the peak strength of intact rock and fractured rock masses, is one of the most widely used strength criteria in rock mechanics and rock engineering (Eberhardt 2012; Hoek and Brown 1980b; Hoek et al. 2000; Wyllie and Mah 2004). The criterion was originally developed using triaxial strength data and for applications in underground excavation design, and was, therefore, expressed in terms of major and minor principal stress (Hoek and Brown 1980a, b), that is

$$\sigma_1 = \sigma_3 + \sqrt{m\sigma_c\sigma_3 + s\sigma_c^2}, \tag{1}$$

or

$$\sigma_1 = \sigma_3 + \sigma_c\sqrt{m\frac{\sigma_3}{\sigma_c} + s}. \tag{2}$$

Here, σ_1 and σ_3 are the major and minor principal stresses at peak strength, σ_c is the uniaxial compressive strength of the intact rock, and m and s are dimensionless empirical constants. This original form of the criterion is most useful in situations where the strength of an element of a rock mass is being assessed in terms of principal stresses (Hoek 1983; Hoek and Brown 1980b, 1988). However, there is considerable interest in applying the criterion to engineering problems such as slope design in heavily jointed rock masses (Hoek 1983; Hoek and Brown 1988; Wyllie and Mah 2004) and shear strength reduction related rock slope stability analyses (Dawson et al. 2000; Hammah et al. 2004), and for this, a formulation of the criterion in terms of shear and normal stresses is more suitable. To this end, several attempts have been made to derive a shear–normal stress form of the criterion. Indeed, the literature contains three seemingly different formulations of the Hoek–Brown criterion in terms of shear and normal stresses: one is the equation derived by Dr John

Bray at Imperial College and reported by Hoek (1983), and two further equations are those developed by Ucar (1986) and Londe (1988) (Brown and Hoek 1988; Hoek 1983; Hoek and Brown 1988, 1997; Londe 1988; Ucar 1986; Wyllie and Mah 2004). These are referred to as the Bray, Ucar, and Londe equations hereafter.

Surprisingly, a review of the literature reveals that, despite these relations being presented many years ago and remaining in common use, no analytical derivation or explicit comparison has been published regarding the equivalence of these three forms of the Hoek–Brown criterion in shear–normal space. In fact, the formulations seem only to have been appraised qualitatively in terms of ease-of-use. For instance, Hoek and Brown (1988) noted that "these three sets of equations are different in appearance and they all yield identical results", but give no proof that the three forms are equivalent. Furthermore, and in case of the Bray equation, which has been considered as the "most convenient for incorporation into computer programs" by Hoek and Brown (1988) and frequently referred to as such by others afterwards (Hoek et al. 2000; Wyllie and Mah 2004), its derivation is missing from the literature.

Although it has been almost 40 years since the Hoek–Brown criterion was first proposed and a new version—the generalised Hoek–Brown criterion (Hoek 1994)—has been introduced later and widely used, the original Hoek–Brown criterion is still important to the rock mechanics community, as is mentioned in Hoek (1994) that "the original criterion has been found to work well for most rocks of good to reasonable quality". In addition, the original Hoek–Brown criterion is helpful for estimating the strength of intact rock (when $s=1$). Particularly, for numerical tools such as the combined finite-discrete element method (FEMDEM) (Munjiza 2004), since its capability to explicitly explore rock mass behaviour from microscopic viewpoints, the strengths of intact rock, rather than those of the rock masses, are usually used as input parameters. Thus, compared with the more complex generalised Hoek–Brown criterion, the analytical shear–normal stress form of the original Hoek–Brown criterion provides an easy-to-implement approach to support these numerical simulations. Moreover, the three shear–normal stress forms of the original Hoek–Brown criterion continue to appear in the literature (e.g. Bertuzzi et al. 2016; Lee and Bobet 2014; Lee and Pietruszczak 2017; Saroglou and Tsiambaos 2008; Senent et al. 2013; Shen et al. 2012), despite no comment being provided regarding their relationship. Therefore, it appears that the relation between the three shear stress-normal stress forms and the original principal stress form of the Hoek–Brown criterion remains an important open question.

With the goal of bringing clarity to this subject, this paper analytically demonstrates the equivalence of the three shear–normal stress forms of the original Hoek–Brown criterion. This formal demonstration of equivalence confirms that the three equations do generate identical results, and means that selection of the form of the shear–normal stress equation of the Hoek–Brown criterion may indeed be based solely on ease-of-use in any particular application. Below, we first present these three equations, show how the Bray equation may be obtained, and then demonstrate their equivalence.

2 Three Shear–Normal Stress Forms of the Hoek–Brown Criterion—the Bray, Ucar, and Londe Equations

2.1 The Bray Equation

When the Hoek–Brown criterion was first proposed in 1980, only an empirical shear strength equation was given in Hoek and Brown (1980a). To assist analyses in those cases where shear failure is dominant, in a later paper—Hoek (1983)—a shear–normal stress form of the criterion was presented, and the derivation accredited to Dr. John Bray at Imperial College. Although the Bray equation is commonly cited in the rock mechanics literature (e.g., Brown and Hoek 1988; Hoek and Brown 1988, 1997; Wyllie and Mah 2004), its derivation seems never to have been presented. We firstly present this equation below, and then give a detailed derivation of it.

The Bray shear–normal stress form of the Hoek–Brown criterion is given by

$$\tau_\alpha = (\cot\beta - \cos\beta)\frac{m\sigma_c}{8}, \tag{3}$$

where τ_α denotes the shear strength corresponding to normal stress σ_α, and β is the instantaneous friction angle at a given value of τ_α and σ_α (point B in Fig. 1). The instantaneous friction angle β can be calculated from

$$\beta = \arctan\left[4h\cos^2\left(30 + \frac{1}{3}\arcsin h^{-3/2}\right) - 1\right]^{-1/2}, \tag{4}$$

where

$$h = 1 + \frac{16(m\sigma_\alpha + s\sigma_c)}{3m^2\sigma_c}. \tag{5}$$

In use, the dimensionless parameter h is calculated from the normal stress σ_α, and from this, the instantaneous friction angle β determined. The corresponding shear strength τ_α is then obtained using Eq. (3). The instantaneous cohesive strength c associated with the point $(\sigma_\alpha, \tau_\alpha)$ is given simply by

$$c = \tau_\alpha - \sigma_\alpha \tan\beta. \tag{6}$$

The Bray equation may be derived using the method introduced by Balmer (1952). To begin, using the triangle BCO of Mohr's circle in Fig. 1, we establish that

$$\left(\sigma_\alpha - \frac{\sigma_1 + \sigma_3}{2}\right)^2 + \tau_\alpha^2 = \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2. \tag{7}$$

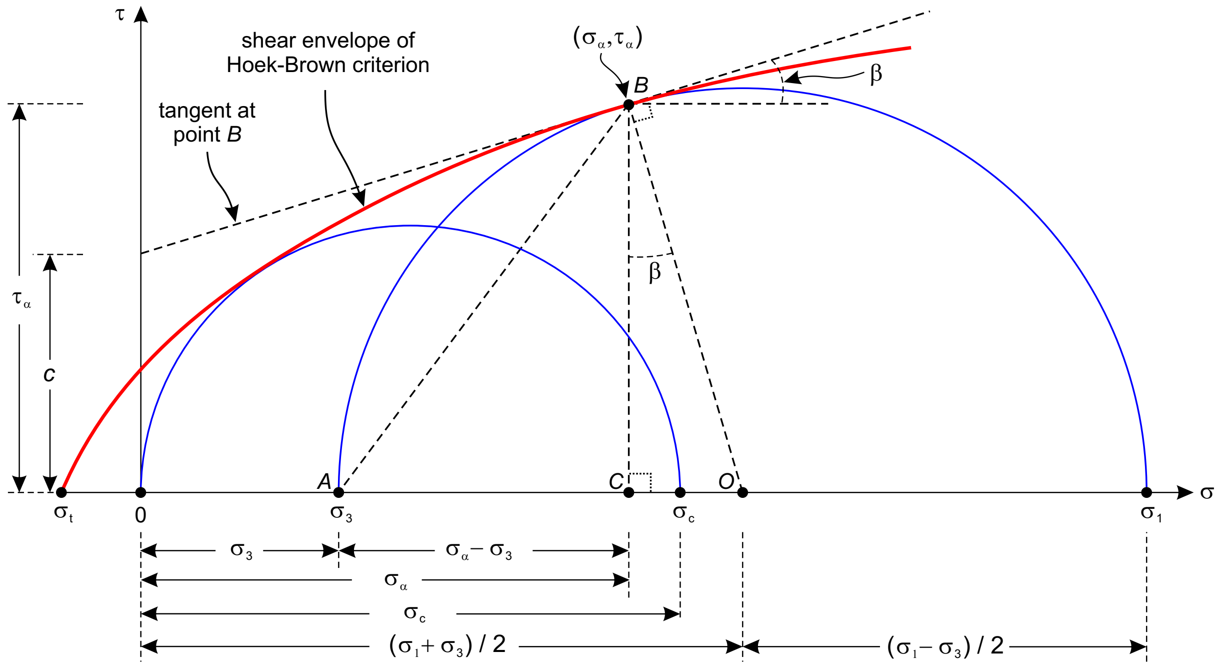

Fig. 1 Graphical representation of the terms in the Bray and Ucar equations (after Ucar 1986)

Taking the partial derivative of σ_1 with respect to σ_3 and solving the resulting equation for σ_α yields

$$\sigma_\alpha = \frac{\sigma_1 - \sigma_3}{\frac{\partial\sigma_1}{\partial\sigma_3} + 1} + \sigma_3. \tag{8}$$

A detailed derivation of Eq. (8) appears in Eqs. (9) to (14) of Ucar (1986). Substituting Eq. (8) into Eq. (7) and solving for τ_α gives

$$\tau_\alpha = \frac{\sigma_1 - \sigma_3}{\frac{\partial\sigma_1}{\partial\sigma_3} + 1}\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}. \tag{9}$$

Note that Eqs. (8) and (9) can alternatively be obtained using the method introduced by Carranza-Torres (2004).

Continuing with triangle BCO in Fig. 1, and using Eqs. (8) and (9), the following trigonometric functions of the friction angle β can be derived:

$$\sin\beta = \frac{\overline{CO}}{\overline{BO}} = \frac{(\sigma_1 + \sigma_3)/2 - \sigma_\alpha}{(\sigma_1 - \sigma_3)/2} = \frac{\frac{\partial\sigma_1}{\partial\sigma_3} - 1}{\frac{\partial\sigma_1}{\partial\sigma_3} + 1}, \tag{10}$$

$$\cos\beta = \frac{\overline{BC}}{\overline{BO}} = \frac{\tau_\alpha}{(\sigma_1 - \sigma_3)/2} = \frac{2\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}}{\frac{\partial\sigma_1}{\partial\sigma_3} + 1}, \tag{11}$$

$$\cot\beta = \frac{\overline{BC}}{\overline{CO}} = \frac{\tau_\alpha}{\frac{\sigma_1+\sigma_3}{2} - \sigma_\alpha} = \frac{2\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}}{\frac{\partial\sigma_1}{\partial\sigma_3} - 1}. \tag{12}$$

Rearranging the Hoek–Brown criterion (Eq. 2) to obtain $(\sigma_1 - \sigma_3)$ and this substituting into Eq. (9), we obtain

$$\tau_\alpha = \frac{\sigma_c\sqrt{m\frac{\sigma_3}{\sigma_c} + s}\cdot\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}}{\frac{\partial\sigma_1}{\partial\sigma_3} + 1}. \tag{13}$$

Multiplying the numerator and denominator of Eq. (13) by $(\partial\sigma_1/\partial\sigma_3 - 1)$ gives

$$\tau_\alpha = \frac{\sigma_c\sqrt{m\frac{\sigma_3}{\sigma_c} + s}\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}\left(\frac{\partial\sigma_1}{\partial\sigma_3} - 1\right)}{\left(\frac{\partial\sigma_1}{\partial\sigma_3} + 1\right)\left(\frac{\partial\sigma_1}{\partial\sigma_3} - 1\right)}. \tag{14}$$

Taking the partial derivative of the Hoek–Brown criterion in Eq. (2) with respect to σ_3 and solving for $(\partial\sigma_1/\partial\sigma_3 - 1)$ gives

$$\frac{\partial\sigma_1}{\partial\sigma_3} - 1 = \frac{m}{2\sqrt{m\frac{\sigma_3}{\sigma_c} + s}}, \tag{15}$$

and substituting Eq. (15) for $(\partial\sigma_1/\partial\sigma_3 - 1)$ in the numerator of Eq. (14), rearranging and making use of the relations given in Eqs. (11) and (12) leads to

$$\tau_\alpha = \frac{m\sigma_c}{2} \frac{\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}}{\left(\frac{\partial\sigma_1}{\partial\sigma_3}+1\right)\left(\frac{\partial\sigma_1}{\partial\sigma_3}-1\right)} = \frac{m\sigma_c}{4}\left(\frac{\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}}{\frac{\partial\sigma_1}{\partial\sigma_3}-1} - \frac{\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}}{\frac{\partial\sigma_1}{\partial\sigma_3}+1}\right) = \frac{m\sigma_c}{8}\left(\frac{2\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}}{\frac{\partial\sigma_1}{\partial\sigma_3}-1} - \frac{2\sqrt{\frac{\partial\sigma_1}{\partial\sigma_3}}}{\frac{\partial\sigma_1}{\partial\sigma_3}+1}\right) = \frac{m\sigma_c}{8}(\cot\beta - \cos\beta). \tag{16}$$

Equation (16) is the Bray equation.

2.2 The Ucar Equation

With the purpose of developing new methods of slope stability analysis, Ucar (1986) presented another solution for the shear strength envelope corresponding to the principal stress form of the Hoek–Brown criterion. The Ucar equation is

$$\tau_\alpha = \frac{m\sigma_c}{8}\frac{1}{\tau'_\alpha\left[\tau'_\alpha\sqrt{1+\tau'^2_\alpha}+\left(1+\tau'^2_\alpha\right)\right]}, \tag{17}$$

where τ'_α is the gradient of the tangent at point $B(\sigma_\alpha, \tau_\alpha)$ on the shear strength envelope (Fig. 1), and is thus equal to $\tan\beta$ of the Bray equation. Note that to allow convenient comparison with the other shear–normal stress equations, Eq. (17) presented here is a re-arrangement of Eq. (42) in Ucar (1986), and the original Ucar equation is

$$\frac{m\sigma_c}{8} = \tau_\alpha\tau'_\alpha\left[\tau'_\alpha\sqrt{1+\tau'^2_\alpha}+\left(1+\tau'^2_\alpha\right)\right]. \tag{18}$$

A detailed derivation of the Ucar equation is presented in Ucar (1986). In addition, in a subsequent discussion paper, Brown and Hoek (1988) compared Ucar's equation to the Bray equation on the basis of ease-of-use for calculating the shear strength, but gave neither a comparison of equivalency nor a discussion of how the two formulations may relate to each other.

2.3 The Londe Equation

In another discussion paper to Ucar (1986), Londe (1988) derived a dimensionless shear–normal stress form of the Hoek–Brown criterion by first dividing both sides of the Hoek–Brown equation by $m\sigma_c$ to give

$$\frac{\sigma_1}{m\sigma_c} = \frac{\sigma_3}{m\sigma_c} + \sqrt{\frac{\sigma_3}{m\sigma_c} + \frac{s}{m^2}}, \tag{19}$$

and then writing this equation in the more compact form

$$N_1 = N_3 + \sqrt{N_3}, \tag{20}$$

where

$$N_1 = \frac{\sigma_1}{m\sigma_c} + \frac{s}{m^2} \tag{21}$$

and

$$N_3 = \frac{\sigma_3}{m\sigma_c} + \frac{s}{m^2}, \tag{22}$$

where N_1 and N_3 are the dimensionless principal stresses, and the Mohr's circle constructed on N_1 and N_3, together with the associated shear strength envelope, is shown in Fig. 2. In this figure,

$$d = N_1 - N_3 = \frac{\sigma_1 - \sigma_3}{m\sigma_c}, \tag{23}$$

and the point P has coordinates

$$N = \frac{\sigma}{m\sigma_c} + \frac{s}{m^2}, \tag{24}$$

and

$$T = \frac{\tau}{m\sigma_c}. \tag{25}$$

The Londe equation is

$$T = \frac{1}{8}(\cot\phi - \cos\phi), \tag{26}$$

where ϕ the slope angle of the line tangent to the Hoek–Brown envelope at point $P(N, T)$ (Fig. 2) is given by

$$\phi = \arcsin\left[2h^{1/2}\cos\left(\frac{1}{3}\arccos(-h)^{-3/2}\right)\right]^{-1}, \tag{27}$$

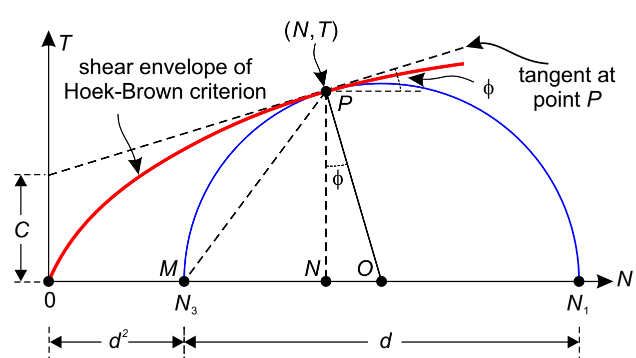

Fig. 2 Graphical representation of the terms in the Londe equation (after Londe 1988)

Table 1 Three shear–normal stress forms of the Hoek–Brown criterion

The Bray equation	$\tau_\alpha = \frac{m\sigma_c}{8}(\cot\beta - \cos\beta)$
The Ucar equation	$\tau_\alpha = \frac{m\sigma_c}{8}\frac{1}{\tau'_\alpha\left[\tau'_\alpha\sqrt{1+\tau'^2_\alpha}+(1+\tau'^2_\alpha)\right]}$
The Londe equation	$T = \frac{1}{8}(\cot\phi - \cos\phi)$

with h being given by

$$h = 1 + \frac{16}{3}N. \tag{28}$$

For any particular normal stress σ, the dimensionless parameter N is calculated from Eq. (24), h and ϕ obtained from Eqs. (28) and (27), respectively, and the dimensionless shear strength T obtained from Eq. (26). Finally, the shear strength is given by $\tau = T \cdot m\sigma_c$ (based on Eq. (25)).

Table 1 summarises the three formulations, and a detailed comparison of these three shear–normal stress forms of Hoek–Brown criterion is presented in the next section.

3 Comparison of the Bray, Ucar, and Londe Equations

3.1 Comparison Between the Bray and Ucar Equations

In the Ucar equation of Eq. (17), τ'_α is the slope of the tangent at the point $B(\tau_\alpha, \sigma_\alpha)$ on the shear envelope in Fig. 1, i.e., $\tau'_\alpha = \tan\beta$. Substituting $\tau'_\alpha = \tan\beta$ into the Ucar equation, we obtain

$$\tau_\alpha = \frac{m\sigma_c}{8}\frac{1}{\tan\beta\left(\tan\beta\sqrt{1+\tan^2\beta}+1+\tan^2\beta\right)}. \tag{29}$$

Then, replacing $\tan\beta$ in Eq. (29) by $\sin\beta/\cos\beta$, the Ucar equation evolves as follows:

$$\begin{aligned}\tau_\alpha &= \frac{m\sigma_c}{8}\frac{1}{\frac{\sin\beta}{\cos\beta}\left(\frac{\sin\beta}{\cos\beta}\sqrt{1+\frac{\sin^2\beta}{\cos^2\beta}}+1+\frac{\sin^2\beta}{\cos^2\beta}\right)}\\ &= \frac{m\sigma_c}{8}\frac{\cos^3\beta}{\sin\beta(\sin\beta+1)}\\ &= \frac{m\sigma_c}{8}(\cot\beta-\cos\beta).\end{aligned} \tag{30}$$

This shows the Ucar equation to be equivalent to the Bray equation, and the latter can be considered as a simplified version of the former. This also supports the statement by Hoek and Brown (1988) reported in Sect. 1, that the Bray equation is the "most convenient".

3.2 Comparison Between the Bray and Londe Equations

To compare the Bray and Londe equations, we first investigate the relation between the angle ϕ of Londe equation (see Fig. 2) and the angle β in both the Bray and Ucar equations (see Fig. 1). Based on the derivation of Bray equation presented in Sect. 2.1, parameters N and T in Fig. 2 can be expressed as

$$N = \frac{N_1+N_3}{2} - \frac{N_1-N_3}{2}\sin\phi, \tag{31}$$

and

$$T = \frac{N_1-N_3}{2}\cos\phi. \tag{32}$$

The tangent of the shear strength envelope (Fig. 2) at point $P(N,T)$ is expressed as

$$T = C + N\tan\phi, \tag{33}$$

where C is the intercept of the tangent on the T-axis. Substituting Eqs. (31) and (32) into Eq. (33) and rearranging for N_1, we obtain

$$N_1 = 2C\frac{\cos\phi}{1-\sin\phi} + N_3\frac{1+\sin\phi}{1-\sin\phi}. \tag{34}$$

Taking the partial derivative of Eq. (34) with respect to N_3, we obtain

$$\frac{\partial N_1}{\partial N_3} = \frac{1+\sin\phi}{1-\sin\phi}, \tag{35}$$

from which we find

$$\sin\phi = \frac{\frac{\partial N_1}{\partial N_3}-1}{\frac{\partial N_1}{\partial N_3}+1}. \tag{36}$$

Taking the partial derivative of Eq. (20) with respect to N_3, we obtain

$$\frac{\partial N_1}{\partial N_3} = 1 + \frac{1}{2\sqrt{N_3}}. \tag{37}$$

Substituting Eq. (37) into Eq. (36) and rearranging gives

$$\sin\phi = \frac{1}{1+4\sqrt{N_3}}, \tag{38}$$

and substituting Eq. (22) into Eq. (38) gives

$$\sin\phi = \frac{1}{1 + 4\sqrt{\frac{\sigma_3}{m\sigma_c} + \frac{s}{m^2}}}. \tag{39}$$

Now, rearranging Eq. (15) and substituting into Eq. (10) leads to

$$\sin\beta = \frac{1}{1 + 4\sqrt{\frac{\sigma_3}{m\sigma_c} + \frac{s}{m^2}}}. \tag{40}$$

Thus, as the right-hand sides of Eqs. (39) and (40) are identical, we see that the angle ϕ (Fig. 2) in Londe equation is equal to the angle β (Fig. 1) in the Bray and Ucar equations, thereby demonstrating that the tangent to the criterion has the same gradient in their respective coordinate spaces. That $\phi = \beta$ allows the Londe equation (Eq. (26)) to be written as

$$T = \frac{1}{8}(\cot\beta - \cos\beta), \tag{41}$$

and given that $\tau = T \cdot m\sigma_c$ (from Eq. (25)), we finally reach the Bray equation based on the Londe equation, that is

$$\tau = \frac{m\sigma_c}{8}(\cot\beta - \cos\beta). \tag{42}$$

This demonstrates the equivalence between the Bray and Londe equation, and that the latter can be considered as a dimensionless version of the former.

4 Conclusion

In the present paper, we have reviewed three shear–normal stress forms of the Hoek–Brown criterion—the Bray, Ucar, and Londe equations—that can be found in the rock mechanics literature, and noted that derivations of both the Bray equation and the equivalence of these equations are missing from the literature. Accordingly, we have given a derivation of the Bray equation, and then, through a series of derivations and comparisons, demonstrated how the Bray, Ucar, and Londe equations are equivalent but presented in different formats: the Bray equation is a simplified version of the Ucar equation, and the Londe equation can be considered as a dimensionless version of the other two. This formal equivalence confirms that the selection of the form of the shear–normal stress equation of Hoek–Brown may be based on ease-of-use in any particular application.

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Compliance with Ethical Standards

Conflict of interest We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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